SIMILAR SOLUTIONS OF BOUNDARY LAYER EQUATIONS IN THE PRESENCE OF A MAGNETIC FIELD

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The author investigates the problem of a plane boundary layer in a viscous incompressible fluid with high conductivity in the presence of a magnetic field. The class of potential flows for which the system examined in [1] reduces to an ordinary system is determined.

In [1] M. V. Belubekyan suggested a method of reducing the nonlinear system of partial differential equations of a boundary layer in the presence of a magnetic field to ordinary differential equations. Belubekyan's method is based on the expansion in powers of x of the parameter characterizing the pressure gradient $\alpha(x) = x(dU/dx)/U$.

Noting that $\alpha(x)$ may have the form

$$\alpha(x) = \frac{a + bx}{1 + cx},$$

we will determine the class of potential motions for which the system considered by Belubekyan reduces to an ordinary system. This method can be extended to the case when $\alpha(x)$ is expressed as the ratio of two polynomials.

Boundary layer equations. As in [2] and [3], we will consider a viscous incompressible conductive fluid with high electrical conductivity. Moreover, we assume that: a) the motion is plane, b) the applied electric field **E** is equal to zero, c) the magnetic field **H** is applied in the plane of motion and at infinity is parallel to the velocity of potential motion.

With these assumptions the equations of the stationary boundary layer take the form [1]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (1-\beta^2)U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{1}{4\pi\rho}\left(H_x\frac{\partial H_x}{\partial x} + H_y\frac{\partial H_x}{\partial y}\right),$$

$$uH_y - vH_x = -\frac{1}{4\pi\sigma}\frac{\partial H_x}{\partial y},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0. \tag{1}$$

System (1) must be integrated with the following boundary conditions:

$$u = v = H_y = 0 \quad \text{for} \quad y = 0,$$

$$u \to U(x), \quad H_x \to \frac{H_0}{U_0}U(x) \quad \text{for} \quad y \to \infty. \tag{2}$$

In these equations the coordinate x is taken in the plane of motion, and the coordinate y normal to that plane.

We set

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x},$$

$$H_x = \frac{\partial A}{\partial y}, \quad H_y = -\frac{\partial A}{\partial x}, \tag{3}$$

and make the change of variables

$$x = x, \quad \eta = y\sqrt{\frac{U}{\nu x}}, \quad f(x, \eta) = \Psi/\sqrt{\nu x U},$$

$$g(x, \eta) = A\Big/\left(\frac{H_0}{U_0}\sqrt{\nu x U}\right). \tag{4}$$

Using the differentiation formulas

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x}\right) + \frac{1}{2}\frac{\alpha - 1}{x}\eta\frac{\partial}{\partial\eta},$$

$$\frac{\partial}{\partial y} = \sqrt{\frac{U}{\nu x}}\frac{\partial}{\partial\eta}, \tag{5}$$

where

$$\alpha(x) = x\frac{dU}{dx}\Big/U, \tag{6}$$

we obtain the following system of equations:

$$2\frac{\partial^3 f}{\partial\eta^3} + f\frac{\partial^2 f}{\partial\eta^2} - \beta^2 g\frac{\partial^2 g}{\partial\eta^2} + \alpha(x)\left\{f\frac{\partial^2 f}{\partial\eta^2} - 2\left(\frac{\partial f}{\partial\eta}\right)^2 + 2 - \beta^2\left[g\frac{\partial^2 g}{\partial\eta^2} - 2\left(\frac{\partial g}{\partial\eta}\right)^2 + 2\right]\right\} =$$

$$= 2x\left[\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial x\partial\eta} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial\eta^2} - \beta^2\left(\frac{\partial g}{\partial\eta}\frac{\partial^2 g}{\partial x\partial\eta} - \frac{\partial g}{\partial x}\frac{\partial^2 g}{\partial\eta^2}\right)\right],$$

$$\frac{2}{\Pr_m}\frac{\partial^2 g}{\partial\eta^2} + f\frac{\partial g}{\partial\eta} - g\frac{\partial f}{\partial\eta} + \alpha(x)\left(f\frac{\partial g}{\partial\eta} - g\frac{\partial f}{\partial\eta}\right) =$$

Table

Forms of U(x) and α(x) for Self-Similar Motions

$U(x)$	$\alpha(x)=x\frac{dU}{dx}\Big/U$	Values of a, b, c
$1-x$	$-\frac{x}{1-x}$	$a=0,\ b=-1,\ c=-1$
$1+x$	$\frac{x}{1+x}$	$a=0,\ b=1,\ c=1$
$(1-x)^n$	$-\frac{nx}{1-x}$	$a=0,\ b=-n,\ c=-1$
$(1+x)^n$	$\frac{nx}{1+x}$	$a=0,\ b=n,\ c=1$
$\frac{1}{1-x}$	$\frac{x}{1-x}$	$a=0,\ b=1,\ c=-1$
$\frac{1}{1+x}$	$-\frac{x}{1+x}$	$a=0,\ b=-1,\ c=1$
$\frac{1}{(1-x)^n}$	$\frac{nx}{1-x}$	$a=0,\ b=n,\ c=-1$
$\frac{1}{(1+x)^n}$	$-\frac{nx}{1+x}$	$a=0,\ b=-n,\ c=1$
e^{-nx}	$-nx$	$a=0,\ b=-n,\ c=0$
e^{nx}	nx	$a=0,\ b=n,\ c=0$
$x-x^2$	$\frac{1-2x}{1-x}$	$a=1,\ b=-2,\ c=-1$
$x+x^2$	$\frac{1+2x}{1+x}$	$a=1,\ b=2,\ c=1$

$$= 2x\left(\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial \eta}\frac{\partial f}{\partial x}\right),$$

$$f = \frac{\partial f}{\partial \eta} = 0, \quad g = 0 \quad \text{for } \eta = 0,$$

$$\frac{\partial f}{\partial \eta} \to 1, \quad \frac{\partial g}{\partial \eta} \to 1 \quad \text{for } \eta \to \infty. \tag{7}$$

Solution of system (7). We will solve system (7), noting that α(x) can be expressed in the form

$$\alpha(x) = \frac{a+bx}{1+cx}. \tag{8}$$

The method by means of which it is possible to select the velocity of potential motion U(x), for which α(x) is expressed in the form (8), and hence for which the motion in the boundary layer is self-similar, is illustrated in the table.

We substitute ratio (8) into system (7) and find its solution in the form of a series in powers of x:

$$f(x, \eta) = f_0(\eta) + xf_1(\eta) + x^2f_2(\eta) + \ldots,$$

$$g(x, \eta) = g_0(\eta) + xg_1(\eta) + x^2g_2(\eta) + \ldots. \tag{9}$$

Then for determining the functions f_0, g_0, f_1, g_1, etc. we obtain the following systems of equations:

$$f_0''' + \frac{a+1}{2}f_0f_0'' - a(f_0'^2 - 1) -$$

$$-\beta^2\left[\frac{a+1}{2}g_0g_0'' - a(g_0'^2 - 1)\right] = 0,$$

$$\frac{1}{\mathrm{Pr}_m}g_0'' + \frac{a+1}{2}(f_0g_0' - g_0f_0') = 0; \tag{10}$$

$$f_1''' + \frac{a+1}{2}f_0f_1'' - (2a+1)f_0'f_1' +$$

$$+\frac{a+3}{2}f_0''f_1 - \beta^2\left[\frac{a+1}{2}g_0g_1'' -\right.$$

$$\left.-(2a+1)g_0'g_1' + \frac{a+3}{2}g_0''g_1\right] =$$

$$= b\left[f_0'^2 - \frac{1}{2}f_0f_0'' - 1 -\right.$$

$$\left.-\beta^2\left(g_0'^2 - \frac{1}{2}g_0g_0'' - 1\right)\right] + ac\left[1 - f_0'^2 + \frac{1}{2}f_0f_0'' -\right.$$

$$\left.-\beta^2\left(1 - g_0'^2 + \frac{1}{2}g_0g_0''\right)\right],$$

$$\frac{1}{\mathrm{Pr}_m}g_1'' + \frac{a+1}{2}(f_0g_1' - g_0f_1') + \frac{a+3}{2}(g_0'f_1 - f_0'g_1) =$$

$$= \frac{b}{2}(f_0'g_0 - g_0'f_0) + \frac{ac}{2}(f_0g_0' - g_0f_0'), \tag{11}$$

. . . .

The boundary conditions for systems (10) and (11) are as follows:

$$f_0 = g_0 = f_1 = g_1 = \ldots = 0, \quad f_0' = f_1' = \ldots = 0$$

$$\text{for } \eta = 0,$$

$$f_0' \to 1, \quad g_0' \to 1, \quad f_1' = g_1' = \ldots = 0 \quad \text{for } \eta \to \infty. \tag{12}$$

We note that system (10) is nonlinear, and its solution depends on the value of a (a = 0 for a flat plate without a pressure gradient and a = 1 for a cylinder). These equations coincide with the equations obtained by Belubekyan [1] for the case of self-similar motion $U(x) = kx^a$. We also note that the zero-order approximations f_0 and g_0 are independent of the constants b and c, but depend on the value of a. Consequently, they are ordinary functions, whereas the functions f_1 and g_1 are not expressed by ordinary functions, since they contain the constants b and c. However, for a certain value of a these functions can be expressed in terms of ordinary functions, if we select the solution of system (11) in the form [4]

$$f_1(\eta) = bf_{11}(\eta) + cf_{12}(\eta),$$

$$g_1(\eta) = bg_{11}(\eta) + cg_{12}(\eta). \tag{13}$$

Thus, from (11), with (13) in mind, we obtain the following system of ordinary inhomogeneous differential equations:

$$f_i''' + \frac{a+1}{2}f_0f_i'' - (2a+1)f_0'f_i' + \frac{a+3}{2}f_0''f_i -$$

$$-\beta^2\left[\frac{a+1}{2}g_0g_i'' - (2a+1)g_0'g_i' +\right.$$

$$\left.+\frac{a+3}{2}g_0''g_i\right] = A_i,$$

$$\frac{1}{\Pr_m} g_i'' + \frac{a+1}{2}(f_0 g_i' - g_0 f_i') +$$

$$+ \frac{a+3}{2}(f_0 g_i + f_i g_0') = B_i; \quad (14)$$

$$f_i(0) = g_i(0) = f_i'(0) = \ldots = 0,$$

$$f_i'(\infty) = g_i'(\infty) = \ldots = 0. \quad (15)$$

Here, the subscript i denotes 11 or 12, and

$$A_{11} + f_0'^2 - \frac{1}{2} f_0 f_0'' - 1 - \beta^2 \left(g_0'^2 - \frac{1}{2} g_0 g_0'' - 1 \right),$$

$$A_{12} =$$

$$= a \left[1 - f_0'^2 + \frac{1}{2} f_0 f_0'' - \beta^2 \left(1 - g_0'^2 + \frac{1}{2} g_0 g_0'' \right) \right],$$

$$B_{11} = \frac{1}{2}(f_0' g_0 - g_0' f_0), \quad B_{12} = \frac{a}{2}(f_0 g_0' - g_0 f_0').$$

It follows from system (14) that the equations for determining the functions f_{11} and g_{11} coincide with the equations obtained by Belubekyan [1]. Since the functions $f_0(\eta)$, $g_0(\eta)$, $f_{11}(\eta)$, $g_{11}(\eta)$, etc. can be tabulated for various values of a, we can determine u, v, H_x, H_y, etc. We have

$$u = U(x)\{f_0'(\eta) + x[bf_{11}'(\eta) + cf_{12}'(\eta)] + \ldots\},$$

$$H_x = H_0 \frac{U(x)}{U_0}\{g_0'(\eta) + x[bg_{11}'(\eta) + cg_{12}'(\eta)] + \ldots\}. \quad (16)$$

NOTATION

u and v are velocity components along x and y; H_x and H_y are components of the magnetic field; ρ is the density; ν is the viscosity; σ is the electrical conductivity of the fluid; U_0 and H_0 are the velocity and magnetic field strength at infinity; U(x) and $(H_0/U_0)U(x)$ are the velocity and magnetic field strength at the outer edge of the boundary layer; ψ is the stream function; A is the component of magnetic field vector potential perpendicular to the plane of motion; $\beta^2 = H_0^2/4\pi\rho U_0^2$ is the Alfvén number; $\Pr_m$ is the magnetic Prandtl number; a, b, c, and k are constants.

REFERENCES

1. M. V. Belubekyan, IFZh, **6**, no. 8, 1963.
2. G. A. Lyubimov, PMM, **26**, no. 5, 1962.
3. A. G. Kulikovskii and G. A. Lyubimov, Magnetohydrodynamics [in Russian], Fizmatgiz, 1962.
4. H. Görtler, J. Math. and Mech., **6**, no. 1, 1957.

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